

MINI-COURSE ON RECENT WORK OF KOLLÁR–TIAN AND TIAN

SUMMARY

The goal of the mini-course is to understand recent work of Tian [Tia25], which has established Colliot-Thélène’s conjecture on the local-global principle for 0-cycles on geometrically rational surfaces over global function fields, by showing the integral Tate conjecture for 1-cycles on 3-folds over finite fields which are separably rationally connected in codimension 1. The proof of the latter relies on work of Kollár–Tian [KT25], which investigates deformations of stable maps of curves and applies to the study of descent of algebraic equivalence of 1-cycles.

The main theorem of the mini-course is [Tia25, Theorem 1.7]. The proof consists of two parts — an arithmetic part and a geometric part — which may respectively be derived from two other important results [Tia25, Theorem 1.20] and [KT25, Theorem 1.7], both of which follow from the analysis of spaces of 1-cycles.

SCHEDULE

We plan to cover the papers [KT25, Tia25] in four two-hour lectures on January 13, 15, 20, 22.

Jan 13 (Suzuki): Main theorem and deformations of stable maps of curves. In the first half, we introduce Colliot-Thélène’s conjecture on the local-global principle for 0-cycles over global function fields, and discuss its reduction, due to Saito, to the integral Tate conjecture for 1-cycles over finite fields. We then state [Tia25, Theorem 1.7], the main theorem of the mini-course. We describe the strategy of addressing the arithmetic and geometric parts of the cycle map, and summarize the idea of the proof, based on the study of deformations of stable maps of curves in [KT25].

In the second half, we prove [KT25, Theorems 1 and 2], which lift algebraic equivalence of the images of finitely many stable maps of curves to deformation equivalence of the stable maps. We also emphasize an arithmetic subtlety in the proof, handled by a result of Poonen.

Jan 15 (Suzuki): Descent of algebraic equivalence of 1-cycles. In the first half, we prove [KT25, Theorem 7] on the descent of algebraic equivalence of 1-cycles on varieties that are separably rationally connected in codimension 1. Along the way, we briefly recall the technique of smoothing a curve by attaching to it a comb of very free rational curves, and explain how to achieve a relative version of this technique, which is important for our purposes.

In the second half, we discuss [Tia25, Theorems 3.10 and 3.11], more technical versions of [KT25, Theorems 1 and 2] which will be used in Day 3 and 4.

Jan 20 (Paranjape): Coniveau and strong coniveau filtrations I.

In the first half, we recap Day 1 and 2, and prove the geometric part of [Tia25, Theorem 1.7]. We then relate the arithmetic part to the concept of coniveau and strong coniveau filtrations $N^c H^*$, $\tilde{N}^c H^*$. More precisely, we briefly recall a series of work [BO21, Voi22, Ben25, SS25] on the two coniveau filtrations, discuss [SS25, Theorem 1.2], and reduce the arithmetic part to [Tia25, Theorem 1.20] which shows $N^{d-2} H^{2d-3} = \tilde{N}^{d-2} H^{2d-3}$ for d -folds that are separably rationally connected in codimension 1.

In the second half, we introduce the Lawson homology for complex varieties. We state [Tia25, Theorem 4.12] and outline the proof. We then deduce [Tia25, Theorem 1.20] over the complex numbers (= [Tia25, Theorem 4.16]).

Jan 22 (Suzuki): Coniveau and strong coniveau filtrations II.

In the first half, we introduce the Chow sheaves and recall some results from the motivic cohomology theory. We state [Tia25, Theorem 5.12] and outline the proof.

In the second half, we deduce [Tia25, Theorem 1.20] over an arbitrary algebraically closed field (= [Tia25, Theorem 5.16]), and conclude the proof of [Tia25, Theorem 1.7].

REFERENCES

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- [SS25] Federico Scavia and Fumiaki Suzuki. Two coniveau filtrations and algebraic equivalence over finite fields. *Algebr. Geom.*, 12(5):701–734, 2025. [2](#)
- [Tia25] Zhiyu Tian. Local-global principle and integral Tate conjecture for certain varieties. *J. Amer. Math. Soc.*, 38(3):703–782, 2025. [1](#), [2](#)
- [Voi22] Claire Voisin. On the coniveau of rationally connected threefolds. *Geom. Topol.*, 26(6):2731–2772, 2022. [2](#)